

# Higher dimensional Thompson groups have Serre's property FA

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## Abstract

The Thompson group  $V$  is a subgroup of the homeomorphism group of the Cantor set  $C$ . Brin [3] defined higher dimensional Thompson groups  $nV$  as generalizations of  $V$ . For each  $n$ ,  $nV$  is a subgroup of the homeomorphism group of  $C^n$ . We prove that  $nV$  has property  $FA$ , and especially the number of ends of  $nV$  is equal to 1. This is a generalization of the corresponding result of Farley [7], who studied the Thompson group  $V = 1V$ .

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## 1 Introduction

Higher dimensional Thompson groups  $nV$  were introduced by Brin in [3] as generalizations of the Thompson group  $V$ . The Thompson group  $V$  is an infinite simple finitely presented group, which is described as a subgroup of the homeomorphism group of the Cantor set  $C$ . Basic facts about  $V$  are found in a paper by Cannon, Floyd and Parry [6].

Brin first studied the case of  $n = 2$  in detail, and showed that  $V$  and  $2V$  are not isomorphic ([3]),  $2V$  is simple ([3]) and  $2V$  is finitely presented ([4]). These properties also hold true for general  $nV$ . The simplicity of  $nV$  was shown by Brin later in [5]. Bleak and Lanoue showed  $n_1V$  and  $n_2V$

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are isomorphic if and only if  $n_1 = n_2$  in [2]. Hennig and Matucci gave a finite presentation for each  $nV$  ([9]).

In this paper we prove that for each  $n$ ,  $nV$  has property FA. This result is the generalization of the corresponding result of Farley [7], who studied  $V$ .

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## 2 Higher dimensional Thompson groups $nV$

In this section, we give the definition of higher dimensional Thompson groups according to Brin's paper [3]. The symbol  $I$  denotes  $[0, 1)$  throughout this paper.

An  $n$ -dimensional rectangle is defined inductively as follows. First,  $I^n$  is a rectangle.

If  $R = [a_1, b_1) \times \cdots \times [a_i, b_i) \times \cdots \times [a_n, b_n)$  is a rectangle, then for all  $i \in \{1, \dots, n\}$ , the “ $i$ -th left half” and “the  $i$ -th right half” defined by

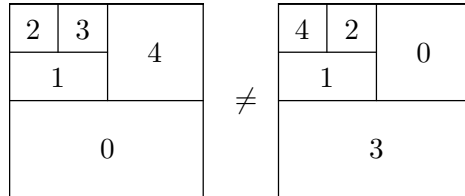
$$R_{l,i} = [a_1, b_1) \times \cdots \times [a_i, (a_i + b_i)/2) \times \cdots \times [a_n, b_n) \quad (2.1)$$

$$R_{r,i} = [a_1, b_1) \times \cdots \times [(a_i + b_i)/2, b_i) \times \cdots \times [a_n, b_n) \quad (2.2)$$

are again rectangles.

Throughout this paper,  $I_l$  denotes  $[0, 1/2) \times I^{n-1}$ . Similarly,  $I_r$  denotes  $[1/2, 1) \times I^{n-1}$ .

Let  $R = [a_1, b_1) \times \cdots \times [a_i, b_i) \times \cdots \times [a_n, b_n)$  be a rectangle. A *corner* of  $R$  is a point in  $\text{cl}(R)$ , whose  $i$ -th coordinate is either  $a_i$  or  $b_i$ . Here  $\text{cl}(R)$  denotes the closure of  $R$  in  $\mathbb{R}^n$ . An  $n$ -dimensional *pattern* is a finite set of  $n$ -dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is  $I^n$ . A *numbered pattern* is a pattern with a one-to-one correspondence to  $\{0, 1, \dots, r-1\}$  where  $r$  is the number of rectangles in the pattern.



From now on, we will identify  $n$ -dimensional rectangle with a subset of  $C^n$  and use the common symbol. First we identify  $I^n$  and  $C^n$ .  $I$  denotes both  $[0, 1)$  and  $C$ . Let  $R$  be a rectangle which is identified with a subset of  $C^n$ ,

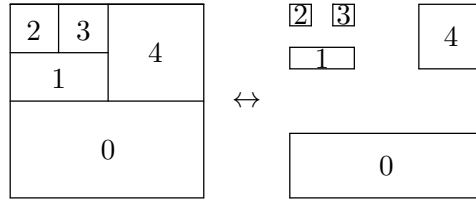
$$R' = C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, b'_i] \times \cdots \times [a'_n, b'_n]. \quad (2.3)$$

Define rectangles  $R_{l,i}$  and  $R_{r,i}$  in the same way as we obtained (2.1) and (2.2). These rectangles are identified respectively with the “ $i$ -th left third” and the “ $i$ -th right third” of  $R'$ , which is defined by

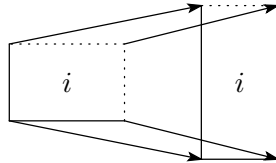
$$C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, (2a'_i + b'_i)/3] \times \cdots \times [a'_n, b'_n], \quad (2.4)$$

$$C^n \cap [a'_1, b'_1] \times \cdots \times [a'_i, (a'_i + 2b'_i)/3] \times \cdots \times [a'_n, b'_n]. \quad (2.5)$$

We proceed by induction. In the same manner, every pattern describes a division of  $C^n$ .

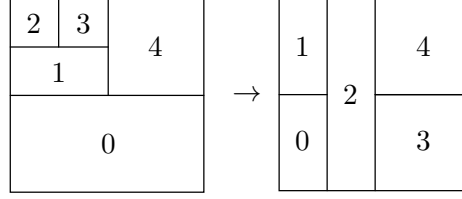


We will construct a self-homeomorphism of  $C^n$  from a pair of numbered patterns with the same number of rectangles. Let  $P = \{P_i\}_{0 \leq i \leq r-1}$  and  $Q = \{Q_i\}_{0 \leq i \leq r-1}$  be numbered patterns. We define  $g(P, Q) : I^n \rightarrow I^n$  which takes each  $P_i$  onto  $Q_i$  affinely so as to preserve the orientation. Namely, the restriction of  $g(P, Q)$  to each  $P_i$  has the form  $(x_1, \dots, x_n) \mapsto (a_1 + 3^{j_1}x_1, \dots, a_n + 3^{j_n}x_n)$  for some integers  $j_1, \dots, j_n$ .



With the former identification of rectangles with subsets of  $C^n$ , above construction defines a self-homeomorphism of  $C^n$ . We again write  $g(P, Q)$  for this homeomorphism.

When  $n = 2$ , we illustrate  $g(P, Q)$  as follows. First we draw  $P$  and  $Q$  as divisions of  $I^2$ . Next we add an arrow from  $P$  to  $Q$ , which indicates the domain and the range.



The  $n$ -dimensional Thompson group  $nV$  is the set of self-homeomorphisms of  $C^n$  of the form  $g(P, Q)$ . Every element of  $nV$  is identified with a partially affine, partially orientation preserving bijection from  $I^n$  to itself.

Next is an important property which will be used in later discussion.

**Theorem 2.1** (Brin [5]). *For all  $n \in \mathbb{N}$ ,  $nV$  is simple.*

### 3 Ends of groups

Let  $\Gamma$  be a path-connected locally finite CW complex. For a compact subset  $K$ ,  $\|\Gamma - K\|$  denotes the number of unbounded connected components of  $\Gamma - K$ . The *number of ends of  $\Gamma$* ,  $e(\Gamma)$ , is defined to be the supremum of  $\|\Gamma - K\|$  taken over all the compact subsets.

When  $\Gamma$  is a graph, we equip  $\Gamma$  with graph metric.  $B(m)$  denotes a ball of radius  $m$  in  $\Gamma$ , based at some fixed vertex. For simplicity, we ignore the dependence of  $B(m)$  on the base point in notation.

Throughout this section,  $G$  denotes a finitely generated group and  $S$  denotes a finite generating set of  $G$ . The *Cayley graph*  $\Gamma_{G,S}$  is a graph whose vertex set is  $G$ , and there is an oriented edge from  $g \in G$  to  $h \in G$  if some  $s \in S$  satisfies  $g \cdot s = h$ .  $G$  acts freely on  $\Gamma_{G,S}$  from the left.

The *number of ends of  $G$* ,  $e(G)$ , is the number of ends of  $\Gamma_{G,S}$ .

**Theorem 3.1** (cf. Geoghegan [8, Corollary 13.5.12]). *Let  $\Gamma$  be a path-connected locally finite CW complex on which  $G$  acts freely. Further suppose that the quotient space  $\Gamma/G$  is a finite CW-complex. Then  $e(\Gamma) = e(G)$ .*

**Proposition 3.2.** (1)  $e(G)$  does not depend on the choice of  $S$ .

(2) (*The Freudenthal-Hopf Theorem*)  $e(G)$  is 0, 1, 2 or  $\infty$ .

(3)  $e(G) = 0$  if and only if  $G$  is finite.

(4)  $e(G) = 2$  if and only if  $G$  has an infinite cyclic subgroup of finite index.

The following result, Stallings' theorem, provides a group-theoretical characterization of the case where  $e(G) \geq 2$ .

**Theorem 3.3** (Stallings [11], Bergman [1]).  *$e(G) \geq 2$  if and only if  $G$  has a structure of an amalgamated product or an HNN-extension on some finite subgroup.*

In the light of this theorem, we can characterize the case of  $e(G) = 1$  in terms of group actions on trees. We say that  $G$  has *property FA* if every simplicial action of  $G$  on a simplicial tree without edge-inversions has a fixed point. Here, a fixed point means  $x \in T$  such that  $g(x) = x$  for every  $g \in G$ .

**Theorem 3.4** (Serre [10]). *If an infinite group has property FA, then  $e(G) = 1$ .*

## 4 $nV$ has property FA

In this section, we prove that  $nV$  has property FA, using a finite presentation of  $nV$ . Throughout this section,  $T$  denotes a simplicial tree. For  $x, y \in T$ , we write  $[x : y]$  for the geodesic joining  $x$  to  $y$ . An action of a group on  $T$  is assumed to be simplicial and to act without edge inversions.

Let  $G$  be a group acting on  $T$ . Let  $g \in G$ . If  $\text{Fix}(g)$  is non-empty,  $g$  is said to be *elliptic*. Otherwise, we say that  $g$  is *hyperbolic*.

The following proposition is a basic fact about group actions on trees.

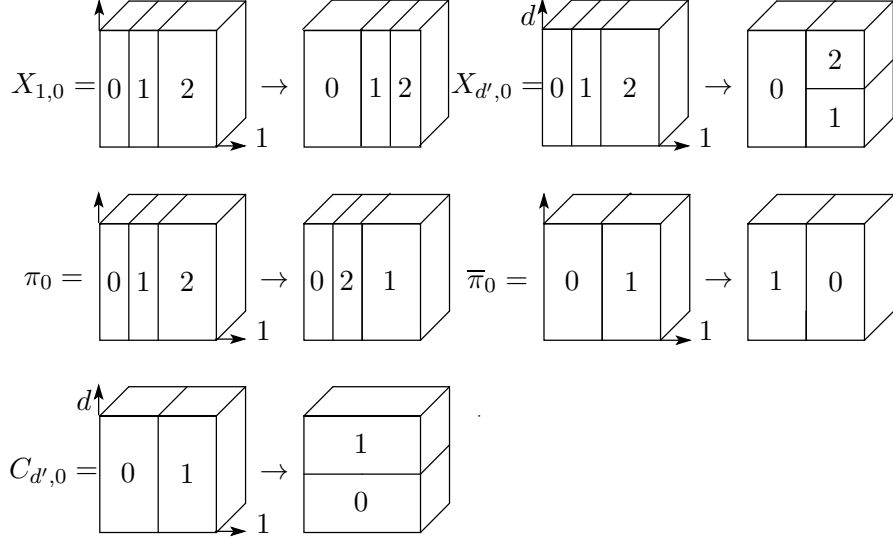
**Proposition 4.1** (Serre [10]). *Let  $G$  be a group acting on  $T$ . Let  $g \in G$ .*

- (1)  $\text{Fix}(g) = \{x \in T \mid g(x) = x\}$  *is either empty or a subtree of  $T$ .*
- (2) *If  $g$  is hyperbolic,  $g$  acts on a unique simplicial line in  $T$  by translation. This line is called the axis of  $g$ .*
- (3) (*Serre's lemma*) *Assume that  $G$  is generated by a finite set of elements  $\{s_j\}_{1 \leq j \leq m}$  such that every element and the multiplication of every two elements are elliptic. Then there is  $x \in T$  which is fixed by every element of  $G$ .*

**Lemma 4.2.** *Let  $G$  be a group acting on  $T$ . If  $g$  and  $h$  are elliptic and satisfy  $gh = hg$ , then  $g$  and  $h$  have a common fixed point.*

*Proof.* Let  $g$  and  $h$  be elliptic elements which satisfy  $gh = hg$ . Assume to the contrary that  $g$  and  $h$  do not have a common fixed point. Fix  $y \in \text{Fix}(h)$ . Let  $[y : x]$  be the shortest geodesic joining  $y$  to  $\text{Fix}(g)$ . The composition of  $g^{-1}([y : x])$  and  $[y : x]$  is  $[y : g^{-1}(y)]$ . Now  $g^{-1}(y) \in \text{Fix}(h)$ , because  $h^{-1}g^{-1}(y) = g^{-1}h^{-1}(y) = g^{-1}(y)$ . By Lemma 4.1 (1),  $[y : g^{-1}(y)] \subset \text{Fix}(h)$ . Therefore  $x \in \text{Fix}(h)$ . This contradicts our assumption.  $\square$

We define  $X_{1,0}, X_{d',0}, C_{d',0}, \pi_0, \bar{\pi}_0 \in nV$  ( $2 \leq d' \leq n$ ) as shown in the following figure. For  $i \geq 1$ ,  $X_{d,i}$  ( $1 \leq d \leq n$ ) is defined inductively. On  $I_r$ ,  $X_{d,i}$  restricts to the identity. For  $x \in I_l$ , we write  $x = (x_1, x_2)$  where  $x_1 \in [0, 1/2)$  and  $x_2 \in I^{n-1}$ . We define  $\phi : I_l \rightarrow I^n$  by  $\phi(x_1, x_2) = (2x_1, x_2)$ . On  $I_l$ ,  $X_{d,i} = X_{d,i-1}\phi$ . Similarly,  $C_{d',i}, \pi_i$  and  $\bar{\pi}_i$  restricts to the identity on  $I_r$  and  $C_{d',i-1}\phi, \pi_{i-1}\phi$  and  $\bar{\pi}_{i-1}\phi$  respectively on  $I_l$ .



**Theorem 4.3** (Hennig and Matucci [9, Theorem 23]). *Let*

$$\Sigma = \{X_{d,i}, C_{d',i}, \pi_i, \bar{\pi}_i\}_{1 \leq d \leq n, 2 \leq d' \leq n, i \geq 0}. \quad (4.1)$$

(1)  $\Sigma$  is a generating set of  $nV$ .

(2) The elements of  $\Sigma$  satisfy the following relations.

$$X_{d'',j}X_{d,i} = X_{d,i}X_{d'',j+1} \quad (i < j, 1 \leq d, d'' \leq n) \quad (4.2)$$

$$C_{d',j}X_{d,i} = X_{d,i}C_{d',j+1} \quad (i < j, 1 \leq d \leq n, 2 \leq d' \leq n) \quad (4.3)$$

$$Y_jX_{d,i} = X_{d,i}Y_{j+1} \quad (i < j, Y \in \{\pi, \bar{\pi}\}, 1 \leq d \leq n) \quad (4.4)$$

$$\pi_jX_{d,i} = X_{d,i}\pi_j \quad (i > j+1, 1 \leq d \leq n) \quad (4.5)$$

$$\pi_jC_{d',i} = C_{d',i}\pi_j \quad (i > j+1, 2 \leq d' \leq n) \quad (4.6)$$

$$\pi_j\pi_i = \pi_i\pi_j \quad (|i-j| > 2) \quad (4.7)$$

$$\bar{\pi}_j\pi_i = \pi_i\bar{\pi}_j \quad (j > i+1) \quad (4.8)$$

$$\bar{\pi}_iX_{1,i} = \pi_i\bar{\pi}_{i+1} \quad (i \geq 0) \quad (4.9)$$

$$C_{d',i}X_{1,i} = X_{d',i}C_{d',i+2}\pi_{i+1} \quad (i \geq 0, 2 \leq d' \leq n) \quad (4.10)$$

$$\pi_iX_{d,i} = X_{d,i+1}\pi_i\pi_{i+1} \quad (i \geq 0, 1 \leq d \leq n) \quad (4.11)$$

**Corollary 4.4.** *Let*

$$S = \{X_{d,1}, X_{d,1}(X_{d,0})^{-1}, C_{d',2}, \pi_0, \pi_3, \bar{\pi}_3\}_{1 \leq d \leq n, 2 \leq d' \leq n}. \quad (4.12)$$

*This is a generating set of  $nV$ .*

$$\begin{aligned} X_{1,1}(X_{1,0})^{-1} &= \begin{array}{c} \uparrow \\ \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline \end{array} \\ \rightarrow 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{|c|c|c|} \hline 0 & 12 & 3 \\ \hline \end{array} \end{array} \\ \\ X_{d',1}(X_{d',0})^{-1} &= \begin{array}{c} \uparrow d \\ \begin{array}{|c|c|c|} \hline 0 & 1 & \begin{array}{c} 3 \\ 2 \end{array} \\ \hline \end{array} \\ \rightarrow 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{|c|c|c|} \hline 0 & \begin{array}{c} 2 \\ 1 \end{array} & 3 \\ \hline \end{array} \end{array} \end{aligned}$$

*Proof.* Let  $\langle S \rangle$  denote a subgroup generated by  $S$ .  $X_{d,0} \in \langle S \rangle$ . For  $i \geq 2$ , the relation (4.2) shows that  $X_{d,i} = (X_{d,0})^{-(i-1)}X_{d,1}(X_{d,0})^{i-1} \in \langle S \rangle$ .

Similarly, the relation (4.4) shows that  $Y_i = (X_{d,0})^{-(i-3)}Y_3(X_{d,0})^{i-3} \in \langle S \rangle$  for  $i \geq 1$ , where  $Y$  is  $\pi$  or  $\bar{\pi}$ . By the relation (4.9),  $\bar{\pi}_0 \in \langle S \rangle$ .

The relation (4.3) shows that  $C_{d',i} = (X_{d,0})^{-(i-2)}C_{d',2}(X_{d,0})^{i-2} \in \langle S \rangle$  for  $i \geq 1$ . By the relation (4.10),  $C_{d',0} \in \langle S \rangle$ .  $\square$

The next lemma is a generalization of Lemma 4.2 in [7].

**Lemma 4.5.** *Let  $g \in nV$  which acts identically on some rectangle. For any action of  $nV$  on a tree  $T$ ,  $g$  is elliptic.*

*Proof.* Let  $g \in nV$  be an element with a rectangle  $R$  on which  $g$  acts as the identity. Assume to the contrary that  $g$  is hyperbolic. We write  $l_g$  for the axis of  $g$ . Let

$$H_g = \{ h \in nV \mid \text{supp}(h) \subseteq R \} \cong nV. \quad (4.13)$$

For every  $h \in H_g$ ,  $hg = gh$  and  $g$  acts on  $h(l_g)$  as a translation. By the uniqueness of the axis,  $h(l_g) = l_g$ . Restricting the action of  $h$  on  $l_g$ , we regard  $h$  as an element of the infinite dihedral group  $D_\infty$ . In this way we obtain a homomorphism  $\Phi : H_g \rightarrow D_\infty$ . By the simplicity of  $H_g$ ,  $\ker \Phi$  is  $H_g$  or the trivial subgroup. We claim that  $\ker \Phi$  is not trivial. Indeed,  $H_g$  has the subgroup which is isomorphic to the Thompson group  $F$ .  $\ker \Phi$  contains the commutator subgroup of  $H_g$ , because every proper quotient of  $F$  is abelian ([6]). Hence  $\ker \Phi = H_g$ .

There is  $k \in nV$  such that  $k \cdot \text{supp}(g) \subseteq R$ . For this  $k$ ,  $kgk^{-1} \in H_g$ . Therefore,  $kgk^{-1}$  is elliptic, which contradicts our assumption that  $g$  is hyperbolic.  $\square$

The following theorem is the main result.

**Theorem 4.6.**  *$nV$  has property FA. Especially,  $e(nV) = 1$ .*

*Proof.* Let  $S$  be the generating set of (4.12). By Serre's lemma, it is enough to show that every element and the product of every two elements of  $S$  are elliptic. By Lemma 4.5, every element of  $S$  is elliptic.

$S = S_1 \cup S_2$ , where

$$S_1 = \{X_{d,1}, C_{d',2}, \pi_3, \bar{\pi}_3\}_{1 \leq d \leq n, 2 \leq d' \leq n}, \quad (4.14)$$

$$S_2 = \{X_{d,1}(X_{d,0})^{-1}, \pi_0\}_{1 \leq d \leq n}. \quad (4.15)$$

Every element of  $S_1$  acts as the identity on  $I_r$ . Every element of  $S_2$  acts as the identity on the "left quarter" of the unit cube,  $[0, 1/4) \times I^{n-1}$ . Therefore, Lemma 4.5 shows that the product of every two elements in  $S_i$  ( $i = 1, 2$ ) is elliptic.

Next we consider  $S_1' = \{C_{d',2}, \pi_3, \bar{\pi}_3\}_{2 \leq d' \leq n} \subset S_1$ . The relations (4.2), (4.3) and (4.4) imply that  $X_{d,1}(X_{d,0})^{-1}$  and  $Z \in S_1'$  are commutative. In fact,

$$X_{d,1}(X_{d,0})^{-1}Z(X_{d,1}(X_{d,0})^{-1})^{-1}Z^{-1} = (X_{d,1}(X_{d,0})^{-1}ZX_{d,0})X_{d,1}^{-1}Z^{-1} = 1.$$

The relations (4.6), (4.7) and (4.8) say that  $\pi_0$  and the elements of  $S_1'$  are commutative. Thus Lemma 4.2 shows the products of  $Z \in S_1'$  and  $X_{d,1}(X_{d,0})^{-1}$  or  $\pi_0$  are elliptic.

The rest of the proof is to show the following lemma.  $\square$



**Lemma 4.7.** *For every  $d, d'' \in \{1, \dots, n\}$ ,*

- (1)  $X_{d,1}$  and  $X_{d'',1}(X_{d'',0})^{-1}$  have a common fixed point.
- (2)  $X_{d,1}$  and  $\pi_0$  have a common fixed point.

*Proof.* (1)  $X_{d'',1}$  and  $X_{d,2}$  act identically on  $I_r$ . By Lemma 4.5 and Serre's lemma, there exists  $y \in T$  which is fixed by  $X_{d'',1}$  and  $X_{d,2}$ .

To obtain a contradiction, suppose  $X_{d,1}$  and  $X_{d'',1}(X_{d'',0})^{-1}$  do not have a common fixed point. Take the shortest geodesic  $[y : x]$  joining  $y$  to  $\text{Fix}(X_{d'',1}(X_{d'',0})^{-1})$ . The composition of  $(X_{d'',1}(X_{d'',0})^{-1})^{-1}[y : x]$  and  $[y : x]$  is  $[y : X_{d'',0}(y)]$ . By the relation (4.2),

$$X_{d,1}X_{d'',0}(y) = X_{d'',0}(X_{d'',0})^{-1}X_{d,1}X_{d'',0}(y) = X_{d'',0}X_{d,2}(y) = X_{d'',0}(y).$$

Therefore,  $[y : X_{d'',0}(y)] \subset \text{Fix}(X_{d,1})$  and  $x \in \text{Fix}(X_{d,1})$ . This contradicts our assumption.

(2) We first show that  $\pi_0$  and  $X_{d,0}$  have a common fixed point. To obtain a contradiction, suppose  $\pi_0$  and  $X_{d,0}$  do not have a common fixed point. By (1),  $\text{Fix}(X_{d,0})$  is not empty. We consider a new element  $(X_{d,0})^{-1}X_{d,1}$ , which acts as the identity on the left one-eighth of  $I^n$ . Since  $\pi_0$  and  $\pi_1$  also act as the identity on this rectangle, we can take  $y$  as a common fixed point of  $(X_{d,0})^{-1}X_{d,1}$ ,  $\pi_0$  and  $\pi_1$ . There is the shortest geodesic  $[y : x]$  joining  $y$  to  $\text{Fix}(X_{d,0})$ . The composition of  $[y : x]$  and  $X_{d,0}([x : y])$  is  $[y : X_{d,0}(y)]$ . By the relation (4.11),

$$\pi_0X_{d,0}(y) = X_{d,1}\pi_0\pi_1(y) = X_{d,1}(y) = X_{d,1}((X_{d,0})^{-1}X_{d,1})^{-1}(y) = X_{d,0}(y).$$

Therefore,  $[y : X_{d,0}(y)] \subset \text{Fix}(\pi_0)$  and  $x \in \text{Fix}(\pi_0)$ . This contradicts our assumption.

We consider a subgroup generated by  $\{\pi_0, (X_{d,0})^{-1}X_{d,1}, X_{d,0}\}$ . By Serre's lemma, this subgroup has a fixed point. Therefore,  $\text{Fix}(\pi_0) \cap \text{Fix}(X_{d,1})$  is not empty.  $\square$

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